

q -Virasoro constraints in matrix models

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ABSTRACT: The Virasoro constraints play the important role in the study of matrix models and in understanding of the relation between matrix models and CFTs. Recently the localization calculations in supersymmetric gauge theories produced new families of matrix models and we have very limited knowledge about these matrix models. We concentrate on elliptic generalization of hermitian matrix model which corresponds to calculation of partition function on $S^3 \times S^1$ for vector multiplet. We derive the q -Virasoro constraints for this matrix model. We also observe some interesting algebraic properties of the q -Virasoro algebra.

KEYWORDS: Matrix Models, Conformal and W Symmetry, Conformal Field Theory

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1 Introduction

The matrix models can be thought of as gauge theories in zero dimensions. The most well-known example is given by the hermitian matrix model. We define the generating function for this model as an integral over hermitian $N \times N$ matrices M

$$Z_N^{\text{herm}}(\{t\}) = \int dM \, e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} \text{Tr}(M^s)}, \quad (1.1)$$

where the measure dM is fixed in such way that it is invariant under the symmetry $M \rightarrow U M U^\dagger$, where U is a $U(N)$ -matrix. Alternatively the matrix model (1.1) can be rewritten as the integral over eigenvalues λ_i

$$Z_N^{\text{herm}}(\{t\}) = \int \prod_{i=1}^N d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} \sum_i \lambda_i^s}, \quad (1.2)$$

where the measure of integration is given by the well-known Vandermonde determinant. This matrix model is well studied and one of the central properties of $Z_N^{\text{herm}}(\{t\})$ is that it is annihilated by an infinite set of differential operator in t_s 's which are known as Virasoro constraints [1, 2].

The natural trigonometric generalization of the hermitian matrix model is defined by the following integral

$$Z_N^{\text{trig}}(\{t\}) = \int \prod_{i=1}^N d\lambda_i \prod_{i<j} \sinh^2(\lambda_i - \lambda_j) e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} \sum_i \lambda_i^s}. \quad (1.3)$$

If we set all the t_s 's to zero except for $s = 2$ then this matrix model describes the partition function for $U(N)$ Chern-Simons theory on S^3 [3, 4]. However we may study the expectation

values of Wilson loops (for a simple knot along the S^1 -fiber) in different representations and thus $Z_N^{\text{trig}}(\{t\})$ can be regarded as the generating function for the expectation values of Wilson loops in different representations (for this simple concrete knot) in Chern-Simons theory. This matrix model can be also derived through the localization technique applied to supersymmetric Chern-Simons theory (supersymmetric vector multiplet in 3D) [5].

The elliptic generalization of hermitian matrix model (1.2) is naturally given by the following formula

$$Z_N^{\text{ell}}(\{t\}) = \oint \prod_{i=1}^N \frac{dz_i}{z_i} \prod_{i<j} \theta\left(\frac{z_i}{z_j}; q\right) \theta\left(\frac{z_j}{z_i}; q\right) e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} \sum_i z_i^s}, \quad (1.4)$$

where the θ functions is defined as follows

$$\theta(z; q) = \prod_{k=0}^{\infty} (1 - zq^k)(1 - z^{-1}q^{k+1}). \quad (1.5)$$

If we set all t_n 's to zero the model (1.4) corresponds to the partition function of a vector multiplet on $S^3 \times S^1$ [6, 7]. If we allow the supersymmetric Wilson loops then $Z_N^{\text{ell}}(\{t\})$ can be thought of as the generating function for the expectation values of Wilson loops in different representations. Alternatively the matrix model (1.4) can be written in other coordinates $z_i = e^{i\lambda_i}$. However the form (1.4) is more standard for the discussion of partition functions and the supersymmetric indices on $S^3 \times S^1$.

The hermitian model $Z_N^{\text{herm}}(\{t\})$ satisfies the Virasoro constraints, see [8] for the review of the subject. The natural question is whether the trigonometric $Z_N^{\text{trig}}(\{t\})$ and elliptic $Z_N^{\text{ell}}(\{t\})$ generalisations also satisfy some type of Virasoro constraints. The goal of this paper is to answer this question. We will show that $Z_N^{\text{trig}}(\{t\})$ satisfies the Virasoro constraints while the elliptic model $Z_N^{\text{ell}}(\{t\})$ satisfies the deformed q -Virasoro constraints. On the way we observe some interesting properties of the q -Virasoro algebra.

The paper is organised as follows: in section 2 we review the derivation of the Virasoro constraints for the hermitian matrix model and we derive the Virasoro constraints for the trigonometric generalization. In section 3 we discuss a different approach to the derivation of Virasoro constraints, we introduce the basics of q -calculus and derive the q -Virasoro constraints for a toy model. In section 4 we apply these ideas to the elliptic generalization of the hermitian matrix model and derive q -Virasoro constraints. Section 5 contains the technical details of the derivation of the q -Virasoro algebra and the discussion of subtleties. We conclude in section 6 and make some general remarks about our results.

2 Virasoro constraints for hermitian matrix model

In this section we review the derivation of the Virasoro constraints in the hermitian matrix model. In our presentation we closely follow the original work [2].

The nature of Virasoro constraints comes from the simple observation that the integral does not change under the change of variables. Let us consider matrix integral

$$Z_N^{\text{herm}}(\{t\}) = \int \prod_i d\lambda_i \prod_{i \neq j} (\lambda_i - \lambda_j) e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} \sum_i \lambda_i^s} \quad (2.1)$$

with the corresponding saddle-point equation

$$\sum_{s \geq 1} \frac{t_s}{s!} s \lambda_i^{s-1} + 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 0. \quad (2.2)$$

Under the shift

$$\lambda_i \rightarrow \lambda_i + \epsilon_n \lambda_i^{n+1}, \quad n \geq -1. \quad (2.3)$$

the effective action in (2.1) changes as the following:

$$\begin{aligned} \delta \left(\sum_{s \geq 0} \frac{t_s}{s!} \sum_i \lambda_i^s + \frac{1}{2} \sum_{i \neq j} \log(\lambda_i - \lambda_j)^2 \right) &= \epsilon_n \sum_i \left(\sum_{s \geq 0} \frac{t_s}{s!} s \lambda_i^{s+n} + \sum_{j \neq i} \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{\lambda_i - \lambda_j} \right) \\ &= \epsilon_n \sum_i \lambda_i^{n+1} \left(\sum_{s \geq 0} \frac{t_s}{s!} s \lambda_i^{s-1} + 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) = 0, \end{aligned} \quad (2.4)$$

where in the last step we have used equations of motion (2.2). Hence (2.3) is the on-shell symmetry of the partition function (2.1). At the same time we can express the invariance of the integral under this symmetry in the form of constraints. To do this we collect the variation linear in ϵ_n under the integral (2.1) which leads to the following expressions

$$\left\langle \sum_{s=1}^{\infty} s \frac{t_s}{s!} \sum_i \lambda_i^{s-1} \right\rangle, \quad n = -1, \quad (2.5)$$

$$\langle N^2 + \sum_{s=0}^{\infty} s \frac{t_s}{s!} \sum_i \lambda_i^s \rangle, \quad n = 0, \quad (2.6)$$

$$\langle (n+1) \sum_i \lambda_i^n + \sum_{i \neq j} \sum_{k=0}^n \lambda_i^k \lambda_j^{n-k} + \sum_{s=0}^{\infty} s \frac{t_s}{s!} \sum_i \lambda_i^{s+n} \rangle, \quad n \geq 1, \quad (2.7)$$

where expectation values are taken with respect to the partition function (2.1). Last expectation value can be rewritten combining the first and second terms leading to

$$\left\langle \sum_{i,j} \sum_{k=0}^n \lambda_i^k \lambda_j^{n-k} + \sum_{s=0}^{\infty} s \frac{t_s}{s!} \sum_i \lambda_i^{s+n} \right\rangle, \quad n \geq 1. \quad (2.8)$$

Expressions (2.5), (2.6) and (2.8) are generated by the following operators acting on $Z_N^{\text{herm}}(\{t\})$

$$\begin{aligned} L_{-1} &= \sum_{k=1}^{\infty} t_k \frac{\partial}{\partial t_{k-1}}, & L_0 &= \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_k} + N^2, \\ L_n &= \sum_{k=0}^n (n-k)! k! \frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum_{k=0}^{\infty} \frac{k(k+n)!}{k!} t_k \frac{\partial}{\partial t_{k+n}}, & n &\geq 1. \end{aligned} \quad (2.9)$$

These are the well-known Virasoro constraints

$$L_n Z_N^{\text{herm}}(\{t\}) = 0, \quad n \geq -1, \quad (2.10)$$

and they satisfy the following algebra

$$[L_n, L_m] = (n - m)L_{n+m} . \quad (2.11)$$

Next let us consider the trigonometric version of the hermitian matrix model

$$Z_N^{\text{trig}}(\{t_s\}) = \int \prod_i d\lambda_i \prod_{i \neq j} \sinh(\beta(\lambda_i - \lambda_j)) e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} \sum_i \lambda_i^s}, \quad (2.12)$$

where we have introduced the deformation parameter β . The model is invariant under the following transformations

$$\lambda_i \rightarrow \lambda_i + \frac{\epsilon_n}{2\beta} e^{2\beta n \lambda_i}, \quad n \geq -1 . \quad (2.13)$$

We collect the variation term linear in ϵ_n under the integral which leads to the following expectation values

$$\begin{aligned} & \langle -N \sum_i e^{-2\beta \lambda_i} + \frac{1}{2\beta} \sum_{s=1}^{\infty} \sum_i s \frac{t_s}{s!} e^{-2\beta \lambda_i} \lambda_i^{s-1} \rangle, \quad n = -1, \\ & \langle \frac{1}{2\beta} \sum_{s=1}^{\infty} \sum_i s \frac{t_s}{s!} \lambda_i^{s-1} \rangle, \quad n = 0, \\ & \langle \sum_{i,j} \sum_{k=0}^{n-1} e^{2\beta k \lambda_i} e^{2\beta(n-k) \lambda_j} + \frac{1}{2\beta} \sum_{s=1}^{\infty} s \frac{t_s}{s!} \sum_i e^{2\beta n \lambda_i} \lambda_i^{s-1} \rangle, \quad n \geq 1 . \end{aligned} \quad (2.14)$$

These terms are generated by the following operator

$$\begin{aligned} L_{-1} &= -N \sum_{k=0}^{\infty} (-2\beta)^k \frac{\partial}{\partial t_k} - \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (-2\beta)^{l-1} t_k \frac{(l+k-1)!}{(k-1)!} \frac{\partial}{\partial t_{l+k-1}}, \\ L_0 &= \frac{1}{2\beta} \sum_{k=1}^{\infty} t_k \frac{\partial}{\partial t_{k-1}}, \\ L_n &= \sum_{k=0}^{n-1} \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} (2\beta k)^s (2\beta(n-k))^l \frac{\partial^2}{\partial t_s \partial t_l} + \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (2\beta)^{l-1} n^l \frac{(l+k-1)!}{(k-1)!} \frac{t_s}{l!} \frac{\partial}{\partial t_{l+k-1}}, \end{aligned} \quad (2.15)$$

which annihilates $Z_N^{\text{trig}}(\{t_s\})$. Using the binomial expansion we can show that these differential operators satisfy the Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} . \quad (2.16)$$

Previously the Virasoro constraints for $Z_N^{\text{trig}}(\{t_s\})$ were discussed in [9], although in a bit less straightforward fashion.

3 Toy model for q -Virasoro constraints

In this section we would like to reflect on the origin of Virasoro symmetry in integrals and then generalize our observations to the case of q -Virasoro symmetry. To do this we will consider some toy examples of matrix models. On the way we will also introduce some basics in q -calculus and necessary combinatorial tools.

If we consider the functions in one variable x then the classical Virasoro algebra has the following well-known representation as first order differential operators

$$L_n = -x^{n+1} \partial_x . \quad (3.1)$$

Alternatively there exists a different representation by the following operators

$$L_n = -(n+1)x^n - x^{n+1} \partial_x = -\partial_x (x^{n+1} \dots) . \quad (3.2)$$

Consider the integral along the real line of a function $f(x)$

$$\int_{-\infty}^{\infty} dx f(x) , \quad (3.3)$$

then, provided the function $f(x)$ is differentiable and decays fast enough at infinity,¹ this integral has the Virasoro symmetries

$$\int_{-\infty}^{\infty} dx L_n f(x) = - \int_{-\infty}^{\infty} dx \partial_x (x^{n+1} f(x)) = 0 . \quad (3.4)$$

However these Virasoro symmetries cannot be converted to any PDEs since the integral is just a number. Instead we can consider the generating function with infinitely many parameters

$$Z^{\text{toy}}(\{t\}) = \int dx x^\alpha e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} x^s} , \quad (3.5)$$

which encodes many different integrals. Then the condition

$$\int dx L_n \left(x^\alpha e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} x^s} \right) = - \int dx \partial_x \left(x^{n+1} x^\alpha e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} x^s} \right) = 0 \quad (3.6)$$

implies the Virasoro constraints $L_n Z^{\text{toy}}(\{t\}) = 0$ where L_n is defined as the following differential operator

$$L_n = (n + \alpha + 1)n! \frac{\partial}{\partial t_n} + \sum_{k=1}^{\infty} \frac{(k+n)!}{(k-1)!} t_k \frac{\partial}{\partial t_{k+n}} , \quad n \geq 0 , \quad (3.7)$$

¹In order for the integral (3.4) to be well defined function $f(x)$ should satisfy $\lim_{|x| \rightarrow \infty} (x^n f(x)) = 0$ for any positive n .

which satisfy the Virasoro algebra. Thus for the case of the hermitian matrix model (2.1) the Virasoro operators (2.9) can be derived by inserting under the integral the following operators

$$L_n = - \sum_i ((n+1)\lambda_i^n + \lambda_i^{n+1} \partial_{\lambda_i}) = - \sum_i \partial_{\lambda_i} (\lambda_i^{n+1} \dots), \quad (3.8)$$

which by themselves generate the Virasoro algebra. The similar trick can be applied to the trigonometric generalization (2.12).

Now let us introduce the basic notions of the q -calculus with q being some complex number. The quantum number n is defined as

$$[n]_q = \frac{q^n - 1}{q - 1} \quad (3.9)$$

and in the limits $q \rightarrow 1$ it becomes just n . The quantum derivative is defined as the following difference operator

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \quad (3.10)$$

Upon the limit $q \rightarrow 1$ the q -derivative D_q becomes the ordinary derivative. The q -derivative satisfies the modified Leibniz rule

$$D_q(f(x)g(x)) = g(qx)D_q f(x) + f(x)D_q g(x) \quad (3.11)$$

and we have

$$D_q x^n = [n]_q x^{n-1}. \quad (3.12)$$

We can define the following q -Virasoro operator

$$T_n^q = -D_q(x^{n+1} \dots), \quad (3.13)$$

which acts on the function of one variable. Alternatively we can define the operators $-x^{n+1}D_q$, but these two definitions lead to the same algebraic properties. We will concentrate on the definition (3.13) since it is the most suitable for the discussion of matrix models. The operators (3.13) satisfy the following relation

$$q^n T_n^q T_m^q - q^m T_m^q T_n^q = ([n]_q - [m]_q) T_{n+m}^q, \quad (3.14)$$

or alternatively we can rewrite this as follows

$$q^{n+1} T_n^q T_m^q - q^{m+1} T_m^q T_n^q = ([n+1]_q - [m+1]_q) T_{n+m}^q. \quad (3.15)$$

In checking these relations we have to use the properties (3.11) and (3.12). Equivalently the commutator of two generators (3.13) can be represented as

$$[T_n^q, T_m^q] = - \sum_{l=1}^{\infty} f_l (T_{n-l}^q T_{m+l}^q - T_{m-l}^q T_{n+l}^q), \quad (3.16)$$

where the coefficients² f_l are chosen such that $\sum_{l=1}^{\infty} f_l q^l = 1$. There is still another relation we can write if we allow to introduce the generators depending on q^2

$$[T_n^q, T_m^q] = q^{-n-m}([n]_q - [m]_q) \left([2]_q T_{n+m}^{q^2} - T_{n+m}^q \right). \quad (3.17)$$

In principle we can go on and generate the whole tower of new operators $T_n^{q^i}$, $i = 1, 2, 3 \dots$ and they will form infinite dimensional Lie algebra. We leave aside the details and other algebraic properties of these generators for the future work [11]. For us it is important to remember that the q -Virasoro generators (3.13) satisfy the algebraic relation (3.17).

The crucial property of D_q is that its integral over the line (even over the half-line $[0, \infty)$) is identically zero

$$\int_{-\infty}^{\infty} dx D_q f(x) = \frac{1}{q-1} \int_{-\infty}^{\infty} dx \frac{f(qx)}{x} - \frac{1}{q-1} \int_{-\infty}^{\infty} dx \frac{f(x)}{x} = 0. \quad (3.18)$$

Thus an integral over the line vanishes upon the insertion of the operators $D_q(x^{n+1} \dots)$ under the integral. Hence if we take the toy generating function (3.5) and insert the operators $D_q(x^{n+1} \dots)$ we get the following set of identities

$$\int dx D_q \left(x^{n+1} x^\alpha e^{\sum_{s=0}^{\infty} \frac{t_s}{s!} x^s} \right) = 0. \quad (3.19)$$

The exponent can be expanded as follows

$$e^{\sum_{s=1}^{\infty} \frac{t_s}{s!} x^s} = \sum_{n=0}^{\infty} B_n(t_1, t_2, \dots, t_n) \frac{x^n}{n!}, \quad (3.20)$$

where B_n is the n th complete Bell polynomial defined as

$$B_n(t_1, t_2, \dots, t_n) = \det \begin{pmatrix} t_1 & \binom{n-1}{1} t_2 & \binom{n-1}{2} t_3 & \binom{n-1}{3} t_4 & \binom{n-1}{4} t_5 & \dots & t_n \\ -1 & t_1 & \binom{n-2}{1} t_2 & \binom{n-2}{2} t_3 & \binom{n-2}{3} t_4 & \dots & t_{n-1} \\ 0 & -1 & t_1 & \binom{n-3}{1} t_2 & \binom{n-3}{2} t_3 & \dots & t_{n-2} \\ 0 & 0 & -1 & t_1 & \binom{n-4}{1} t_2 & \dots & t_{n-3} \\ 0 & 0 & 0 & -1 & t_1 & \dots & t_{n-4} \\ 0 & 0 & 0 & 0 & -1 & \dots & t_{n-5} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & t_1 \end{pmatrix}, \quad (3.21)$$

where $\binom{n}{k}$ stands for the binomial coefficient $\frac{n!}{k!(n-k)!}$. The complete Bell polynomials are weighted homogeneous polynomials. For instance, the first few Bell polynomials are:

$$\begin{aligned} B_1(t_1) &= t_1, \\ B_2(t_1, t_2) &= t_1^2 + t_2, \\ B_3(t_1, t_2, t_3) &= t_1^3 + 3t_1 t_2 + t_3, \\ B_4(t_1, t_2, t_3, t_4) &= t_1^4 + 6t_1^2 t_2 + 4t_1 t_3 + 3t_2^2 + t_4, \end{aligned} \quad (3.22)$$

²This form of deformed Virasoro algebra was introduced in [10] with prescribed f_l . One can show that the realization (3.13) is a special case of their deformation [11].

and we assume that $B_0 = 1$. We will use the following properties of the Bell polynomials

$$B_l((\alpha + \beta)t_1, \dots, (\alpha + \beta)t_l) = \sum_{p=0}^l \binom{l}{p} B_{l-p}(\alpha t_1, \dots, \alpha t_{l-p}) B_p(\beta t_1, \dots, \beta t_p), \quad (3.23)$$

and

$$\frac{\partial}{\partial t_l} B_n(t_1, \dots, t_n) = \begin{cases} 0, & n < l \\ \frac{n!}{(n-l)!l!} B_{n-l}(t_1, \dots, t_{n-l}), & n \geq l \end{cases}, \quad (3.24)$$

and

$$B_l(\tilde{t}_1, \dots, \tilde{t}_l) = \sum_{p=0}^l q^p \binom{l}{p} B_p(t_1, \dots, t_p) B_{l-p}(-t_1, \dots, -t_{l-p}), \quad (3.25)$$

where $\tilde{t}_k = (q^k - 1)t_k$. These properties are easily derivable from the definition (3.20).

After applying the definition (3.10) and recombining the terms we arrive to the constraints

$$T_n^q Z^{\text{toy}}(\{t\}) = 0, \quad n \geq 0, \quad (3.26)$$

where the operators T_n^q are defined as follows

$$T_n^q = \frac{1}{q-1} \left[\sum_{l=0}^{\infty} \sum_{k=0}^l \frac{[n+l-k+\alpha+1]_q}{(l-k)!k!} B_{l-k}(t_1, t_2, \dots, t_{l-k}) B_k(-t_1, -t_2, \dots, -t_k) \frac{\partial}{\partial t_{n+l}} - n! \frac{\partial}{\partial t_n} \right]. \quad (3.27)$$

Alternatively using the property (3.25) we can rewrite the operator (3.27) as

$$T_n^q = [n + \alpha + 1]_q n! \frac{\partial}{\partial t_n} + \frac{q^{n+\alpha+1}}{q-1} \sum_{k=1}^{\infty} \frac{(k+n)!}{k!} B_k(\tilde{t}_1, \dots, \tilde{t}_k) \frac{\partial}{\partial t_{k+n}}, \quad (3.28)$$

with $\tilde{t}_k = (q^k - 1)t_k$. One can observe that the operator (3.28) collapses to the operator (3.7) in the classical limit $q \rightarrow 1$. We have checked explicitly that the operators T_n^q satisfy the algebra (3.17). In checking the algebra (3.17) we had to use the properties (3.23) and (3.24).

4 q -Virasoro for elliptic hermitian matrix model

In this section we derive the q -Virasoro constraints for the elliptic generalisation of the matrix model. Although any matrix integral will vanish under the insertion of D_q only in the case of the elliptic matrix model the operator D_q talks nice to the matrix model measure.

The partition function for the $4d \mathcal{N}=1$ $U(N)$ gauge theory on $S^3 \times S^1$

$$\int \prod_{i=1}^N d\lambda_i \prod_{i < j} \left(1 - e^{i(\lambda_i - \lambda_j)}\right) \left(1 - e^{i(\lambda_j - \lambda_i)}\right) \prod_{n=1}^{\infty} \left(1 - q^n e^{i(\lambda_i - \lambda_j)}\right)^2 \left(1 - q^n e^{i(\lambda_j - \lambda_i)}\right)^2, \quad (4.1)$$

where the integration is over the real line and $q \equiv e^\beta$ with β being the circumference of S^1 . Performing the change of variables $e^{i\lambda_i} = z_i$ we arrive at the following form of the partition function

$$\oint \prod_{i=1}^N \frac{dz_i}{z_i} \prod_{i < j} \theta\left(\frac{z_i}{z_j}; q\right) \theta\left(\frac{z_j}{z_i}; q\right), \quad (4.2)$$

where the integration is now over the contour around the origin and θ function is defined as in (1.5). Next, we can introduce the generating function

$$Z_N^{\text{ell}}(\{t\}) = \oint \prod_{i=1}^N \frac{dz_i}{z_i} \prod_{i < j} \theta\left(\frac{z_i}{z_j}; q\right) \theta\left(\frac{z_j}{z_i}; q\right) e^{\sum_{k=0}^{\infty} \frac{t_k}{k!} \sum_{i=1}^N z_i^k}, \quad (4.3)$$

for the expectation values of supersymmetric Wilson loop in different representations. Our goal is to show that $Z_N^{\text{ell}}(\{t\})$ is annihilated by the q -Virasoro constraints.

Following the logic from the previous section we can define the differential operator

$$T_n^q = - \sum_{l=1}^N D_q^{z_l} (z_l^{n+1} \dots), \quad (4.4)$$

which acts on the functions of N variables and $D_q^{z_l}$ is q -derivative with respect to the z_l -variable. These operators satisfy the q -Virasoro algebra (3.14)–(3.17). The insertion of these operators under the contour integral (4.3) gives us identically zero. Now we have to analyse in details how these differential operators act on the integrand. Using the properties of θ function

$$\theta(qz; q) = \theta(z^{-1}; q), \quad \theta(q^{-1}z; q) = q^{-1}z^2 \theta(z^{-1}; q), \quad (4.5)$$

we arrive at the following relation

$$\begin{aligned} \sum_{l=1}^N D_q^{z_l} \left(f(z_l) \prod_{i < j} \theta\left(\frac{z_i}{z_j}; q\right) \theta\left(\frac{z_j}{z_i}; q\right) \right) = \\ \sum_{l=1}^N \frac{1}{(q-1)z_l} \left(\frac{f(qz_l)}{f(z_l)} \prod_{j \neq l} q^{-1} \frac{z_j^2}{z_l^2} - 1 \right) f(z_l) \prod_{i < j} \theta\left(\frac{z_i}{z_j}; q\right) \theta\left(\frac{z_j}{z_i}; q\right). \end{aligned} \quad (4.6)$$

For the calculation of the q -derivative of the exponential factor we use the expansion (3.20) in terms of the Bell polynomials B_k

$$\begin{aligned} \exp\left(\sum_{k=1}^{\infty} \frac{t_k}{k!} q^k z_i^k\right) &= \sum_{k=0}^{\infty} \frac{1}{k!} B_k(t_1, \dots, t_k) q^k z_i^k \\ &= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{k! p!} B_k(t_1, \dots, t_k) B_p(-t_1, \dots, -t_p) q^k z_i^{k+p} \exp\left(\sum_{l=1}^{\infty} \frac{t_l}{l!} z_i^l\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} B_k(\tilde{t}_1, \dots, \tilde{t}_k) x^k \exp\left(\sum_{l=1}^{\infty} \frac{t_l}{l!} z_i^l\right). \end{aligned} \quad (4.7)$$

Applying the formulas (4.6) and (4.7) we find that the insertion of the operator T_n^q (4.4) under the integral (4.3) is equivalent to the insertion of the following terms under the integral

$$\frac{1}{q-1} \left[\prod_{j=1}^N z_j^2 \sum_{l=1}^N \sum_{k,p=0}^{\infty} q^{n+1+k-N} \frac{1}{k!p!} B_k(t_1, \dots, t_k) B_p(-t_1, \dots, -t_p) z_l^{k+p+n-2N} - \sum_{l=1}^N z_l^n \right]. \quad (4.8)$$

Thus the expectation value of these terms should be zero. Now our final goal is to generate these terms by taking the appropriate t -derivatives of the integrand of (4.3). In order to do it we need to rewrite $\prod_{i=1}^N z_i$ in terms of sums $\sum_{i=1}^N z_i^k$. This can be done using the Newton's identities

$$\prod_{i=1}^N z_i = \frac{1}{N!} \begin{vmatrix} p_1 & 1 & 0 & \dots \\ p_2 & p_1 & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ p_{N-1} & p_{N-2} & \dots & \dots & p_1 & N-1 \\ p_N & p_{N-1} & \dots & \dots & p_2 & p_1 \end{vmatrix}, \quad (4.9)$$

where $p_k \equiv \sum_{i=1}^N z_i^k$. The terms $\sum_{i=1}^N z_i^k$ can be generated by taking the t -derivatives and thus we can introduce the following differential operator

$$\mathcal{D}_N = \frac{1}{N!} \begin{vmatrix} 2! \frac{\partial}{\partial t_2} & 1 & 0 & \dots \\ 4! \frac{\partial}{\partial t_4} & 2! \frac{\partial}{\partial t_2} & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (2N-2)! \frac{\partial}{\partial t_{2N-2}} & (2N-4)! \frac{\partial}{\partial t_{2N-4}} & \dots & \dots & 2! \frac{\partial}{\partial t_2} & N-1 \\ (2N)! \frac{\partial}{\partial t_{2N}} & (2N-2)! \frac{\partial}{\partial t_{2N-2}} & \dots & \dots & 4! \frac{\partial}{\partial t_4} & 2! \frac{\partial}{\partial t_2} \end{vmatrix}, \quad (4.10)$$

with the property that

$$\prod_{j=1}^N z_j^2 e^{\sum_{k=0}^{\infty} \frac{t_k}{k!} \sum_{i=1}^N z_i^k} = \mathcal{D}_N \left(e^{\sum_{k=0}^{\infty} \frac{t_k}{k!} \sum_{i=1}^N z_i^k} \right). \quad (4.11)$$

Combining all together we obtain the following q -Virasoro operator

$$T_n^q = \frac{1}{q-1} \left[\sum_{k,p=0}^{\infty} q^{n+1+k-N} \frac{(k+p+n-2N)!}{k!p!} B_k(t_1, \dots, t_k) B_p(-t_1, \dots, -t_p) \right. \\ \left. \times \mathcal{D}_N \frac{\partial}{\partial t_{k+p+n-2N}} - n! \frac{\partial}{\partial t_n} \right], \quad (4.12)$$

which annihilates the generating function $Z_N^{\text{ell}}(\{t\})$. Using the property (3.25) we can rewrite the operator (4.12) as follows

$$T_n^q = \frac{1}{q-1} \left[q^{n+1-N} \sum_{l=0}^{\infty} \frac{(l+n-2N)!}{l!} B_l(\tilde{t}_1, \dots, \tilde{t}_l) \mathcal{D}_N \frac{\partial}{\partial t_{l+n-2N}} - n! \frac{\partial}{\partial t_n} \right]. \quad (4.13)$$

We see that generically the operators T_n^q are higher order differential operators and action of these operators on $Z_N^{\text{ell}}(\{t\})$ generates the insertion of the terms (4.8) under the integral. It is crucial to stress that there are many different higher order operators which will generate exactly the same insertion (4.8). This fact complicates the calculation of the algebra and we elaborate more on this point later. In the next section we will show that the operators (4.13) satisfy the following algebra

$$[T_n^q, T_m^q] = q^{-n-m}([n]_q - [m]_q) \left([2]_q T_{n+m}^{q^2} - T_{n+m}^q \right), \quad (4.14)$$

where the operators $T_n^{q^2}$ are defined below in (4.18). The operators $T_n^{q^2}$ annihilate $Z_N^{\text{ell}}(\{t\})$. Indeed if we continue to calculate the algebra we will get the whole tower of operators $T_n^{q^j}$, $j = 1, 2, 3, \dots$ which annihilate $Z_N^{\text{ell}}(\{t\})$.

In order to define explicitly the operators $T_n^{q^2}$ we have to insert under the integral (4.3) the following difference operator

$$T_n^{q^2} = - \sum_{l=1}^N D_{q^2}^{z_l} (z_l^{n+1} \dots). \quad (4.15)$$

Using the following properties of the θ function

$$\theta(q^2 z; q) = -q^{-1} z^{-1} \theta(z^{-1}; q), \quad \theta(q^{-2} z; q) = -q^{-3} z^3 \theta(z^{-1}; q), \quad (4.16)$$

we obtain following Ward identities

$$\left\langle \frac{1}{q^2 - 1} \left[\prod_{j=1}^N z_j^4 \sum_{l=1}^N \sum_{p=0}^{\infty} q^{2n+4-4N} \frac{1}{p!} B_p(\hat{t}_1, \dots, \hat{t}_p) z_l^{p+n-4N} - \sum_{l=1}^N z_l^n \right] \right\rangle = 0, \quad (4.17)$$

where by $\langle \dots \rangle$ we mean the insertion of this expression under the integral (4.3). These Ward identities can be expressed in the form $T_n^{q^2} Z_N^{\text{ell}}(\{t\}) = 0$, where the differential operator $T_n^{q^2}$ is given by

$$T_n^{q^2} = \frac{1}{q^2 - 1} \left[q^{2n+4-4N} \sum_{l=0}^{\infty} \frac{(l+n-4N)!}{l!} B_l(\hat{t}_1, \dots, \hat{t}_l) \tilde{D}_N \frac{\partial}{\partial t_{l+n-4N}} - n! \frac{\partial}{\partial t_n} \right]. \quad (4.18)$$

Here \tilde{D}_N is the differential operator defined as

$$\tilde{D}_N = \frac{1}{N!} \begin{vmatrix} 4! \frac{\partial}{\partial t_4} & 1 & 0 & \dots \\ 8! \frac{\partial}{\partial t_8} & 4! \frac{\partial}{\partial t_4} & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (4N-4)! \frac{\partial}{\partial t_{4N-4}} & (4N-8)! \frac{\partial}{\partial t_{4N-8}} & \dots & \dots & 4! \frac{\partial}{\partial t_4} & N-1 \\ (4N)! \frac{\partial}{\partial t_{4N}} & (4N-4)! \frac{\partial}{\partial t_{4N-4}} & \dots & \dots & 8! \frac{\partial}{\partial t_8} & 4! \frac{\partial}{\partial t_4} \end{vmatrix}. \quad (4.19)$$

Again we see that the operators $T_n^{q^2}$ are higher order differential operators. Analogously we can define the operators $T_n^{q^j}$.

In the expression (4.12) there is a problem since the operator T_n^q is defined only for the case $n \geq 2N$. Similar problems exist for the operator $T_n^{q^2}$ which is defined for the case $n \geq 4N$. To resolve this problem we can insert the different q -Virasoro operator $-\sum_{l=1}^N D_q^{z_l} \left(z_l^{2N(n+1)+1} \dots \right)$ under the integral. This leads to the following differential operator

$$\tilde{T}_n^q = \sum_{p \geq 0} q^{2N(n+1)+1-N} \frac{(p+2Nn)!}{p!} B_k(\tilde{t}) \mathcal{D}_N \frac{\partial}{\partial t_{p+2Nn}} - (2Nn+2N)! \frac{\partial}{\partial t_{2N(n+1)}}, \quad (4.20)$$

so that the operator is well defined for any $n \geq -1$. The algebra can be calculated in completely analogous way.

5 Calculation of the algebra

In this section we derive the algebra (4.14) for the operators T_n^q defined in (4.13) and the operators $T_n^{q^2}$ defined in (4.18). The operators T_n^q and $T_n^{q^2}$ are higher order differential operators which generate concrete insertions under the integral. However different differential operators can generate exactly the same insertions under the integral. Thus in principle we can have different representations for T_n^q and $T_n^{q^2}$ as higher order differential operators in t 's. We have to keep in mind this feature of these operators.

Our goal is to check the relation (4.14) for the operators (4.13). For this we calculate the commutator $[T_n^q, T_m^q]$. In order to calculate it we need to know the action of the operator \mathcal{D}_N on the complete Bell polynomials. One can easily derive the following relation

$$\mathcal{D}_N \left(e^{\sum_{s=1}^{\infty} \frac{t_s}{s!} x^s} \right) = 0, \quad (5.1)$$

which upon the expansion (3.20) implies that the operator \mathcal{D}_N annihilates the Bell polynomials

$$\mathcal{D}_N B_k(t_1, \dots, t_k) = 0. \quad (5.2)$$

Next we calculate the following relation

$$\mathcal{D}_N \left(e^{\sum_{s=1}^{\infty} \frac{\tilde{t}_s}{s!} x^s} \right) = (-1)^{N+1} (q^2 - 1) x^{2N} e^{\sum_{s=1}^{\infty} \frac{\tilde{t}_s}{s!} x^s}, \quad (5.3)$$

where we recall that $\tilde{t}_s = (q^s - 1)t_s$. Expanding this formula in x we get the following action of \mathcal{D}_N

$$\mathcal{D}_N B_k(\tilde{t}_1, \dots, \tilde{t}_k) = \begin{cases} 0, & k < 2N \\ (-1)^{N+1} (q^2 - 1) \frac{k!}{(k-2N)!} B_{k-2N}(\tilde{t}_1, \dots, \tilde{t}_{k-2N}), & k \geq 2N \end{cases} \quad (5.4)$$

Finally we need the following relation

$$B_p(\hat{t}_1, \dots, \hat{t}_p) = \sum_{k \leq p} \frac{p!}{k!(p-k)!} q^k B_k(\tilde{t}_1, \dots, \tilde{t}_k) B_p(\tilde{t}_1, \dots, \tilde{t}_p), \quad (5.5)$$

where $\hat{t}_k \equiv (q^k + 1)\tilde{t}_k = (q^{2k} - 1)t_k$. This relation follows directly from the expansion (3.20). Using these formulas we can straightforwardly calculate $[T_n^q, T_m^q]$. However for the case of arbitrary number N of eigenvalues this problem is complicated and the corresponding expressions are enormous. Thus we will concentrate on the simplest cases of $N = 2$ and $N = 3$.

Using the above formulas it is straightforward to evaluate commutator of T_n^q operators for the case $N = 2$

$$\begin{aligned}
 [T_n^q, T_m^q] = & \frac{q^{-n-m}}{(q-1)^2} ([n]_q - [m]_q) \left[q^{2n+2m-6} \sum_{p \geq 0} \frac{1}{p!} B_p(\hat{t}) \mathcal{D}_2 \left(\mathcal{D}_2(p+n+m-8)! \frac{\partial}{\partial t_{p+n+m-8}} - \right. \right. \\
 & (q^2 - 1)(p+n+m-4)! \frac{\partial}{\partial t_{p+n+m-4}} + (q^2 - 1)2!(p+n+m-6)! \frac{\partial^2}{\partial t_2 \partial t_{p+n+m-6}} \Big) - \\
 & \left. q^{n+m-1} \sum_{p \geq 0} \frac{(p+n+m-4)!}{p!} B_p(\tilde{t}) \mathcal{D}_2 \frac{\partial}{\partial t_{p+n+m-4}} \right]. \quad (5.6)
 \end{aligned}$$

For the case of $N = 3$ we obtain

$$\begin{aligned}
 [T_n^q, T_m^q] = & \frac{q^{-n-m}}{(q-1)^2} ([n]_q - [m]_q) \left[q^{2n+2m-10} \sum_{p \geq 0} \frac{1}{p!} B_p(\hat{t}) \mathcal{D}_3 \left(\mathcal{D}_3(p+n+m-12)! \frac{\partial}{\partial t_{p+n+m-12}} + \right. \right. \\
 & (q^2 - 1)(p+n+m-6)! \frac{\partial}{\partial t_{p+n+m-6}} - \\
 & \frac{1}{2}(q^2 - 1) \left(4! \frac{\partial}{\partial t_4} - (2!)^2 \frac{\partial^2}{\partial t_2^2} \right) (p+n+m-10)! \frac{\partial}{\partial t_{p+n+m-10}} - \\
 & \left. (q^2 - 1)2!(p+n+m-8)! \frac{\partial^2}{\partial t_2 \partial t_{p+n+m-8}} \right) - \\
 & \left. q^{n+m-2} \sum_{p \geq 0} \frac{(p+n+m-6)!}{p!} B_p(\tilde{t}) \mathcal{D}_3 \frac{\partial}{\partial t_{p+n+m-6}} \right]. \quad (5.7)
 \end{aligned}$$

The last terms in both (5.6) and (5.7) contributes to T_{n+m}^q operator in (4.14). However to completely match these expressions with the commutation relations (4.14) we need the explicit form (4.18) of the operator $T_n^{q^2}$. Notice that $\tilde{\mathcal{D}}_N \sim (\mathcal{D}_N)^2$, where \sim means that these two operators are equivalent upon the action on the partition function (4.3). As we can see the expressions in (5.6) and (5.7) are complicated and do not match the operator (4.18). However if we act with the right hand side of (5.6) on the partition function we obtain the familiar terms. This action can be obtained directly by the substitution

$$\frac{\partial}{\partial t_a} \rightarrow (x_1^a + x_2^a), \quad \mathcal{D} \rightarrow x_1^2 x_2^2. \quad (5.8)$$

Then the operators written in the first two lines of (5.6) result in the following expectation value:

$$\begin{aligned}
 & \frac{([n]_q - [m]_q)}{(q-1)} \langle q^{n+m-6} \sum_{p \geq 0} \frac{1}{p!} B_p(\hat{t}) x_1^2 x_2^2 ((x_1^{a-8} + x_2^{a-8}) x_1^2 x_2^2 - (q^2 - 1)(x_1^{a-4} + x_2^{a-4}) \\
 & + (q^2 - 1)(x_1^{a-6} + x_2^{a-6})(x_1^2 + x_2^2)) \rangle = \frac{([n]_q - [m]_q)}{(q-1)} \langle q^{n+m-4} \sum_{p \geq 0} \frac{1}{p!} B_p(\hat{t}) x_1^4 x_2^4 (x_1^{a-8} + x_2^{a-8}) \rangle \\
 & = ([n]_q - [m]_q) \left[[2]_q T_{n+m}^{q^2} + n! \frac{\partial}{\partial t_n} \right] Z, \tag{5.9}
 \end{aligned}$$

where for shortness we have introduced $a = p + n + m$. Combining these terms with the last term in (5.6)

$$q^{n+m-1} \sum_{p \geq 0} \frac{(p + n + m - 4)!}{p!} B_p(\tilde{t}) \mathcal{D}_2 \frac{\partial}{\partial t_{p+n+m-4}} = T_{n+m}^q + n! \frac{\partial}{\partial t_n} \tag{5.10}$$

we can arrive to the desired commutation relation (4.14). One can perform similar calculation for the case $N = 3$ by making the substitution

$$\frac{\partial}{\partial t_a} \rightarrow (x_1^a + x_2^a + x_3^a), \quad \mathcal{D} \rightarrow x_1^2 x_2^2 x_3^2. \tag{5.11}$$

After some simple algebra one can show that the commutation relation (5.7) is compatible with (4.14) once we act on the partition function (4.3).

We have obtained the desired algebra (4.14) but on the way we had to perform some additional manipulations. Let us provide the general explanation for what we did. For this purpose we consider general matrix model of the form

$$Z(\{t\}) = \int d^N x f(x_1, \dots, x_N) e^{\sum_{k=0}^{\infty} \frac{t_k}{k!} \sum_{i=1}^N x_i^k}, \tag{5.12}$$

where the t 's are parameters (either finite or infinite number of them). We are interested to discuss the symmetries of this integral, namely the differential operators D in t 's which annihilate $Z(\{t\})$

$$DZ(\{t\}) = \int d^N x f(x_1, \dots, x_N) \sigma_D(x_1, \dots, x_N) e^{\sum_{k=0}^{\infty} \frac{t_k}{k!} \sum_{i=1}^N x_i^k} = 0. \tag{5.13}$$

The operator D upon acting on the exponent generates the function σ_D . However there is no unique correspondence between D and the function σ_D . Two different operators D and \tilde{D} generate the same function σ_D if

$$(D - \tilde{D}) e^{\sum_{k=0}^{\infty} \frac{t_k}{k!} \sum_{i=1}^N x_i^k} = 0. \tag{5.14}$$

The operators which annihilate the exponent form an ideal among the differential operators. When we study the algebra of the symmetries we have to quotient by this ideal. For the symmetries D_1 and D_2 we have

$$[D_1, D_2]Z(\{t\}) = 0. \tag{5.15}$$

All the symmetries will form a Lie algebra which is the Lie algebra of the operators annihilating Z modulo the ideal discussed above. In calculation of the algebra (4.14) we had to use some identification (5.14). Notice that this is a generic feature of the matrix models and even the simple Virasoro operator (2.9) may have a different representations upon these identifications.

6 Summary

The Virasoro constraints are important for the understanding of the hermitian matrix model. In this work we looked at the elliptic generalization of the hermitian matrix model and we have derived the q -Virasoro constraints for this model. Our deformation of the Virasoro algebra is based on the realization in terms of q -derivatives within the q -calculus and this deformation is a special case of a more general elliptic deformation of the Virasoro algebra. The deformation of Virasoro algebras has been first discussed by Curtright and Zachos in [12] (see also [13] for further explanation and the relevant references). Later the deformation of the Virasoro algebra was introduced in [10, 14, 15] in a different context. There were numerous works on the study of these deformations including different physical realizations of q -Virasoro algebra ([16–22, 25, 26]). However the realization of the general elliptic deformation of Virasoro algebra in terms of concrete difference operators is still missing. Moreover, many algebraic aspects remain a mystery. For example, the realization of the algebra in (3.17) was forced upon us by the matrix model and came as total surprise.

The trigonometric and elliptic deformations of the Virasoro algebra play a crucial role in the higher dimensional gauge theories, as an example see the recent papers [23] and [24] on the role of the elliptic deformation. What we have observed in this paper is just a tip of the iceberg. We think that the q -Virasoro constraints are a generic feature of the partition functions on $S^3 \times S^1$ for different gauge theories. We believe that the elliptic deformation of the Virasoro algebra should appear when one tries to generalise the present analysis to a wider class of theories. However in order to make a further progress we need to understand better the algebraic property of the deformed Virasoro algebra and to find the realization in terms of the concrete difference operators. We plan to answer these questions in a forthcoming work [11].

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